

The predual and John-Nirenberg inequalities on generalized BMO martingale spaces

Yong Jiao, Anming Yang, Lian Wu and Rui Yi
Central South University

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Abstract

In this paper we introduce the generalized BMO martingale spaces by stopping time sequences, which enable us to characterize the dual spaces of martingale Hardy-Lorentz spaces $H_{p,q}^s$ for $0 < p \leq 1, 1 < q < \infty$. Moreover, by duality we obtain a John-Nirenberg theorem for the generalized BMO martingale spaces when the stochastic basis is regular. We also extend the boundedness of fractional integrals to martingale Hardy-Lorentz spaces.

1 Introduction

Basing mainly on the duality, John-Nirenberg inequality and something else, the space BMO (Bounded Mean Oscillation; see [6], [7] and [12]) played a remarkable role in classical analysis and probability. We refer to the monographs [3] and [20] for the function space version, respectively to the monographs [2], [8] and [18] for the martingale version of those theorems.

This paper deals with the John-Nirenberg inequalities and dualities in the martingale theory. Before describing our main results, we recall the classical John-Nirenberg inequalities in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_n\}_{n \geq 0}$ be a nondecreasing sequence of sub- σ -algebras of \mathcal{F} such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We also call $\{\mathcal{F}_n\}_{n \geq 0}$ a stochastic basis (with convention $\mathcal{F}_{-1} = \mathcal{F}_0$). The expectation

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Corresponding email: jiaoyong@csu.edu.cn

operator and the conditional expectation operators relative to \mathcal{F}_n are denoted by \mathbb{E} and \mathbb{E}_n , respectively. The stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is said to be regular, if there exist an absolute constant $R > 0$ such that

$$f_n \leq R f_{n-1}, \quad (1.1)$$

holds for all nonnegative martingales $f = (f_n)_{n \geq 0}$.

A sequence $f = (f_n)_{n \geq 0}$ of random variables such that f_n is \mathcal{F}_n -measurable is said to be a martingale if $\mathbb{E}(|f_n|) < \infty$ and $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \geq 0$. For the sake of simplicity, we assume $f_0 = 0$. For $1 \leq r < \infty$, the Banach spaces BMO_r are defined as follows:

$$BMO_r = \{f = (f_n)_{n \geq 0} \in L_r : \|f\|_{BMO_r} = \sup_n \|(\mathbb{E}_n |f - f_{n-1}|^r)^{\frac{1}{r}}\|_\infty < \infty\}.$$

Here f in $|f - f_{n-1}|^r$ means f_∞ . The usual BMO norm corresponds to $r = 2$ above, i.e., $\|f\|_{BMO} = \|f\|_{BMO_2}$. The John-Nirenberg theorem says that in the sense of equivalent norms,

$$BMO_r = BMO, \quad (1 \leq r < \infty). \quad (1.2)$$

A duality argument yields that (1.1) can be rewritten as follows

$$\|f\|_{BMO} \approx \sup_n \sup_{A \in \mathcal{F}_n} \mathbb{P}(A)^{-\frac{1}{r}} \left(\int_A |f - f_{n-1}|^r d\mathbb{P} \right)^{\frac{1}{r}}. \quad (1.3)$$

Here and in the sequel, $A \approx B$ means that there exist two absolute constants C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

The special contribution of this paper is to define the following generalized BMO martingale space $BMO_{r,q}(\alpha)$ by stopping time sequences.

Definition 1.1. For $1 \leq r, q < \infty, \alpha \geq 0$, the generalized BMO martingale space is defined by

$$BMO_{r,q}(\alpha) = \left\{ f \in L_r : \|f\|_{BMO_{r,q}(\alpha)} = \sup_{k \in \mathbb{Z}} \frac{\sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{r}} \|f - f^{\nu_k}\|_r}{\left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{1+\alpha})^q \right)^{\frac{1}{q}}} < \infty \right\},$$

where the supremum is taken over all stopping time sequences $\{\nu_k\}_{k \in \mathbb{Z}}$ such that $\{2^k \mathbb{P}(\nu_k < \infty)^{1+\alpha}\}_{k \in \mathbb{Z}} \in \ell_q$.

Then the generalized John-Nirenberg theorem, one of our main results, reads as follows.

Theorem 1.2. Suppose that the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and $1 \leq q < \infty$. Then

$$BMO_{r,q}(\alpha) = BMO_{2,q}(\alpha), \quad (1.4)$$

in the sense of equivalent norms for all $1 \leq r < \infty$.

We now explain the relation between (1.1) and (1.4). Let \mathcal{T} be the set of all stopping times relative to $\{\mathcal{F}_n\}_{n \geq 0}$. On one hand, if the stopping time sequence $\{\nu_k\}_{k \in \mathbb{Z}}$ reduces to a sequence whose one element is a stopping time ν and the others are ∞ , then the generalized BMO space $BMO_{r,q}(\alpha)$ reduces to the following Lipschitz space

$$BMO_r(\alpha) = \{f \in L_r : \|f\|_{BMO_r(\alpha)} = \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}-\alpha} \|f - f^\nu\|_r < \infty\}.$$

On the other hand, if the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, it is not very difficult to check that (1.2) can further be reformulated as

$$\|f\|_{BMO} \approx \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}} \|f - f^\nu\|_r. \quad (1.5)$$

See also [22] for the facts above. Hence if $\alpha = 0$, (1.4) exactly implies (1.5). Consequently, (1.1) can be deduced from (1.4) when the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular.

We now turn to the second aim of this paper. The generalized BMO martingale space $BMO_{r,q}(\alpha)$ defined in this paper enable us to characterize the dualities of martingale Hardy-Lorentz spaces for $0 < p \leq 1, 1 < q < \infty$. It is well known that the dual spaces of Lebesgue spaces L_p or Lorentz spaces $L_{p,q}$ are trivial when $0 < p < 1$ (see for instance [5] or [9]), namely,

$$(L_p)^* = (L_{p,q})^* = \{0\}, \quad (0 < p < 1, 0 < q \leq \infty).$$

However, the dual spaces of martingale Hardy spaces are very different from those of Lebesgue spaces L_p and Lorentz spaces $L_{p,q}$. This can be illustrated by the fact that the dual spaces of L_p and $L_{p,q}$ ($0 < p < 1$) are trivial while

$$(H_p^s)^* = BMO_2(\alpha), \quad (0 < p < 1, \alpha = \frac{1}{p} - 1),$$

where H_p^s denotes the martingale Hardy space associated with the conditional quadratic variation, that is,

$$H_p^s = \left\{ f = (f_n)_{n \geq 0} : \|f\|_{H_p^s} = \left\| \left(\sum_{i=1}^{\infty} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}} \right\|_p < \infty \right\}.$$

We refer to [8], [14] and [22] for the fact above. At the same time, Weisz [22] also proved the following duality result for martingale Hardy-Lorentz spaces,

$$(H_{p,q}^s)^* = H_{p',q'}^s, \quad (1 < p < \infty, 1 \leq q < \infty),$$

where p' and q' denote the conjugate numbers of p and q respectively; see Section 2 for definition of $H_{p,q}^s$. But the question how to characterize the dual spaces of martingale Hardy-Lorentz spaces for $0 < p \leq 1, 0 < q < \infty$ is still open. We prove that the dual space of martingale Hardy-Lorentz space is the same as the one of martingale Hardy spaces when $0 < p, q \leq 1$, while it needs the new notion $BMO_{r,q}(\alpha)$ when $0 < p \leq 1, 1 < q < \infty$. In Section 4 we shall show

Theorem 1.3. *Let $0 < p \leq 1$, $\alpha = \frac{1}{p} - 1$. Then*

$$(H_{p,q}^s)^* = BMO_2(\alpha), \quad 0 < q \leq 1;$$

and

$$(H_{p,q}^s)^* = BMO_{2,q}(\alpha), \quad 1 < q < \infty.$$

This paper will be divided into five further sections. In the next section, some notations and basic knowledge will be introduced. In Section 3, the atomic decompositions of martingale Hardy-Lorentz spaces are formulated. In Section 4, using atomic decompositions in Section 3, we prove some dual theorems of martingale Hardy-Lorentz spaces. By duality, the new John-Nirenberg theorem for the generalized BMO martingale space is proved in Section 5. In the final Section, the boundedness of fractional integrals on martingale Hardy-Lorentz spaces are investigated.

In this paper, the set of integers and the set of nonnegative integers are always denoted by \mathbb{Z} and \mathbb{N} , respectively. We use C to denote the absolute constant which may vary from line to line.

2 Notations and preliminaries

We first introduce the distribution function and the decreasing rearrangement. Let f be a measurable function defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the distribution function of f by

$$\lambda_s(f) = \mathbb{P}(\{\omega \in \Omega : |f(\omega)| > s\}), \quad (s \geq 0).$$

And denote by $\mu_t(f)$ the decreasing rearrangement of f , defined by

$$\mu_t(f) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \quad (t \geq 0),$$

with the convention that $\inf \emptyset = \infty$.

We list some properties of distribution functions and decreasing rearrangements in the following proposition. The properties will be used in the proof of theorems in the later sections.

Proposition 2.1. *Let f and g be two measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$, then we have*

- (1) if $|f| \leq |g|$ \mathbb{P} -a.e. then $\lambda_s(f) \leq \lambda_s(g)$ for all $s \geq 0$;
- (2) $\lambda_{s_1+s_2}(f+g) \leq \lambda_{s_1}(f) + \lambda_{s_2}(g)$ for all $s_1, s_2 \geq 0$;
- (3) $\mu_t(af) = |a|\mu_t(f)$ for all $a \in \mathbb{C}$ and $t \geq 0$;
- (4) if $|f| \leq |g|$ \mathbb{P} -a.e. then $\mu_t(f) \leq \mu_t(g)$ for all $t \geq 0$;
- (5) $\mu_{t_1+t_2}(f+g) \leq \mu_{t_1}(f) + \mu_{t_2}(g)$ for all $t_1, t_2 \geq 0$.

The Lorentz space $L_{p,q}(\Omega, \mathcal{F}, \mathbb{P})$, $0 < p < \infty, 0 < q \leq \infty$, consists of those measurable functions f with finite norm or quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty \left(t^{\frac{1}{p}} \mu_t(f) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad (0 < q < \infty),$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} \mu_t(f), \quad (q = \infty).$$

It will be convenient for us to use an equivalent definition of $\|f\|_{p,q}$, namely

$$\|f\|_{p,q} = \left(q \int_0^\infty \left(t \mathbb{P}(|f(x)| > t)^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad (0 < q < \infty),$$

$$\|f\|_{p,\infty} = \sup_{t>0} t \mathbb{P}(|f(x)| > t)^{\frac{1}{p}}, \quad (q = \infty).$$

We recall that Lorentz spaces $L_{p,q}$ increase as the second exponent q increases, and decrease as the first exponent p increases (the second exponent q is not involved). Namely, $L_{p,q_1} \subset L_{p,q_2}$ for $0 < p < \infty$ and $0 < q_1 \leq q_2 \leq \infty$, $L_{p_1,q_1} \subset L_{p_2,q_2}$ for $0 < p_2 < p_1 < \infty$ and $0 < q_1, q_2 \leq \infty$. It is also well known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, then $\|\cdot\|_{p,q}$ is equivalent to a norm. However, for the other values of p and q , $\|\cdot\|_{p,q}$ is only a quasi-norm. In particular, if $0 < q \leq 1$ and $q \leq p < \infty$, then $\|\cdot\|_{p,q}$ is equivalent to a q -norm. Hölder's inequality for Lorentz spaces is the following

$$\|fg\|_{p,q} \leq C \|f\|_{p_1,q_1} \|g\|_{p_2,q_2},$$

where $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

We now introduce martingale Hardy-Lorentz spaces. Denote by \mathcal{M} the set of all martingales $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$ such that $f_0 = 0$. For $f \in \mathcal{M}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ ($n \geq 0$, with convention $f_{-1} = 0$). Then the maximal function, the quadratic variation and the conditional quadratic variation of a martingale f are respectively defined by

$$f_n^* = \sup_{0 \leq i \leq n} |f_i|, \quad f^* = \sup_{n \geq 0} |f_n|,$$

$$S_n(f) = \left(\sum_{i=1}^n |d_i f|^2 \right)^{\frac{1}{2}}, \quad S(f) = \left(\sum_{i=1}^\infty |d_i f|^2 \right)^{\frac{1}{2}},$$

$$s_n(f) = \left(\sum_{i=1}^n \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}, \quad s(f) = \left(\sum_{i=1}^\infty \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}.$$

Let Λ be the collection of all sequences $(\lambda_n)_{n \geq 0}$ of nondecreasing, nonnegative and adapted functions, set $\lambda_\infty = \lim_{n \rightarrow \infty} \lambda_n$. For $f \in \mathcal{M}$, $0 < p < \infty, 0 < q \leq \infty$, let

$$\Lambda[Q_{p,q}](f) = \{(\lambda_n)_{n \geq 0} \in \Lambda : S_n(f) \leq \lambda_{n-1} (n \geq 1), \lambda_\infty \in L_{p,q}\},$$

$$\Lambda[D_{p,q}](f) = \{(\lambda_n)_{n \geq 0} \in \Lambda : |f_n| \leq \lambda_{n-1} (n \geq 1), \lambda_\infty \in L_{p,q}\}.$$

We define martingale Hardy-Lorentz spaces as follows. For $0 < p < \infty, 0 < q \leq \infty$,

$$H_{p,q}^* = \{f \in \mathcal{M} : \|f\|_{H_{p,q}^*} = \|f^*\|_{p,q} < \infty\},$$

$$H_{p,q}^S = \{f \in \mathcal{M} : \|f\|_{H_{p,q}^S} = \|S(f)\|_{p,q} < \infty\},$$

$$H_{p,q}^s = \{f \in \mathcal{M} : \|f\|_{H_{p,q}^s} = \|s(f)\|_{p,q} < \infty\},$$

$$Q_{p,q} = \{f \in \mathcal{M} : \|f\|_{Q_{p,q}} = \inf_{(\lambda_n)_{n \geq 0} \in \Lambda[Q_{p,q}](f)} \|\lambda_\infty\|_{p,q} < \infty\},$$

$$D_{p,q} = \{f \in \mathcal{M} : \|f\|_{D_{p,q}} = \inf_{(\lambda_n)_{n \geq 0} \in \Lambda[D_{p,q}](f)} \|\lambda_\infty\|_{p,q} < \infty\}.$$

If taking $p = q$ in the definitions above, we get the usual martingale Hardy spaces. In order to describe the duality theorems, we need to introduce the Lipschitz space $BMO_r(\alpha)$. For $1 \leq r < \infty, \alpha \geq 0$, the Lipschitz space are defined as follows

$$BMO_r(\alpha) = \{f \in L_r : \|f\|_{BMO_r(\alpha)} < \infty\},$$

where

$$\|f\|_{BMO_r(\alpha)} = \sup_{n \in \mathbb{N}} \sup_{A \in \mathcal{F}_n} \mathbb{P}(A)^{-\frac{1}{r}-\alpha} \left(\int_A |f - \mathbb{E}_n f|^r d\mathbb{P} \right)^{\frac{1}{r}}.$$

Let \mathcal{T} be the set of all stopping times relative to $\{\mathcal{F}_n\}_{n \geq 0}$. For a martingale $f = (f_n)_{n \geq 0} \in \mathcal{M}$ and a stopping time $\nu \in \mathcal{T}$, we denote the stopped martingale by $f^\nu = (f_n^\nu)_{n \geq 0} = (f_{n \wedge \nu})_{n \geq 0}$. Then it is easy to show that

$$\|f\|_{BMO_r(\alpha)} = \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}-\alpha} \|f - f^\nu\|_r, \quad (1 \leq r < \infty, \alpha \geq 0).$$

The main new notion of the present paper is the generalized BMO martingale space $BMO_{r,q}(\alpha)$ ($1 \leq r, q < \infty, \alpha \geq 0$), see Section 1 for the definition. In Definition 1.1, if the stopping time sequence $\{\nu_k\}_{k \in \mathbb{Z}}$ reduces to a sequence whose one element is a stopping time ν and the others are ∞ , then the generalized BMO martingale space $BMO_{r,q}(\alpha)$ reduces to the Lipschitz martingale space $BMO_r(\alpha)$. Obviously, $BMO_{r,q}(\alpha)$ is a subspace of $BMO_r(\alpha)$ and $\|f\|_{BMO_r(\alpha)} \leq \|f\|_{BMO_{r,q}(\alpha)}$.

We will present the atomic decomposition theorems for martingale Hardy-Lorentz spaces in the next section. Now let us introduce the notion of atoms; see for example [22].

Definition 2.2. *A measurable function a is called a (p, ∞) -atom of the first category (or of the second category, or of the third category) if there exists a stopping time $\nu \in \mathcal{T}$ (ν is called the stopping time associated with a) such that*

(i) $a_n = \mathbb{E}_n(a) = 0$, (if $\nu \geq n$), (ii) $\|s(a)\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$ (or (i') $\|S(a)\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$, or (i'') $\|a^\|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}$, respectively).*

These three category atoms are briefly called $(1, p, \infty)$ -atom, $(2, p, \infty)$ -atom and $(3, p, \infty)$ -atom, respectively.

We conclude this section by two lemmas which are very useful to verify that a function is in Lorentz spaces $L_{p,q}$, which are respectively from Lemma 1.1 and Lemma 1.2 in [1].

Lemma 2.3. *Let $0 < p < \infty, 0 < q \leq \infty$, assume that the nonnegative sequence $\{\mu_k\}$ satisfies $\{2^k \mu_k\} \in l^q$. Further suppose that the nonnegative function φ verifies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0 , we have $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} is essentially bounded and satisfies $\|\psi_{k_0}\|_\infty \leq C 2^{k_0}$, and*

$$2^{k_0 \varepsilon p} \mathbb{P}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^{k \varepsilon} \mu_k)^p.$$

Then $\varphi \in L_{p,q}$ and $\|\varphi\|_{p,q} \leq C \|\{2^k \mu_k\}\|_{l_q}$.

Lemma 2.4. *Let $0 < p < \infty$, and let the nonnegative sequence $\{\mu_k\}$ be such that $\{2^k \mu_k\} \in l^q, 0 < q \leq \infty$. Further, suppose that the nonnegative function φ satisfies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0 , we have $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} satisfy*

$$2^{k_0 p} \mathbb{P}(\psi_{k_0} > 2^{k_0})^\varepsilon \leq C \sum_{k=-\infty}^{k_0} (2^k \mu_k^\varepsilon)^p, \quad 0 < \varepsilon < \min(1, \frac{q}{p}),$$

$$2^{k_0 \varepsilon p} \mathbb{P}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^{k \varepsilon} \mu_k)^p.$$

Then $\varphi \in L_{p,q}$ and $\|\varphi\|_{p,q} \leq C \|\{2^k \mu_k\}\|_{l_q}$.

3 Atomic decompositions

The method of atomic decompositions plays an important role in martingale theory; see for instance [10], [13], [15], [16], [21] and [22]. In particular, Jiao, Peng and Liu [11] proved the atomic decompositions of martingale Hardy-Lorentz spaces in 2009. Since $\|\cdot\|_{p,q}$ is equivalent to a q -norm just when $0 < q \leq 1$ and $q \leq p < \infty$, there is a restrictive condition for the converse part of Theorem 2.1 in [11]. We improve Theorem 2.1 in [11] by using the technical Lemma 2.3 and shows that the converse part of Theorem 2.1 in [11] is true for all $0 < p < \infty, 0 < q \leq \infty$.

Theorem 3.1. *If $f = (f_n)_{n \geq 0} \in H_{p,q}^s$ ($0 < p < \infty, 0 < q \leq \infty$), then there exists a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(1, p, \infty)$ -atoms and a sequence $(\mu_k)_{k \in \mathbb{Z}} \in l_q$ of real numbers satisfying $\mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}$ (where A is a positive constant and ν_k is the stopping time associated with a^k) such that*

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k, \quad \text{a.e.,} \quad n \in \mathbb{N}, \quad (3.1)$$

and

$$\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C \|f\|_{H_{p,q}^s}.$$

Conversely, if the martingale f has the above decomposition, then $f \in H_{p,q}^s$ and

$$\|f\|_{H_{p,q}^s} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q},$$

where the infimum is taken over all the above decompositions.

Proof. Assume that $f \in H_{p,q}^s$ ($0 < p < \infty, 0 < q \leq \infty$). For each $k \in \mathbb{Z}$, the stopping time is defined as follows

$$\nu_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}, \quad (\inf \emptyset = \infty).$$

Obviously, the sequence of these stopping times is non-decreasing. Similarly to the proof of Theorem 2.2 in [22] (or see the proof of Theorem 2.1 in [11]), we have

$$\sum_{k \in \mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) = f_n.$$

Let

$$\mu_k = 3 \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}, \quad a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

If $\mu_k = 0$, we assume that $a_n^k = 0$. Then for any fixed $k \in \mathbb{Z}$, $a^k = (a_n^k)_{n \geq 0}$ is a martingale. Considering the stopped martingale $f^{\nu_k} = (f_n^{\nu_k})_{n \geq 0} = (f_{n \wedge \nu_k})_{n \geq 0}$, we have $s(f^{\nu_k}) = s_{\nu_k}(f) \leq 2^k$, $s(f^{\nu_{k+1}}) \leq 2^{k+1}$. Then

$$s(a^k) \leq \frac{s(f^{\nu_{k+1}}) + s(f^{\nu_k})}{\mu_k} \leq \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}},$$

which implies that $(a_n^k)_{n \geq 0}$ is a L_2 -bounded martingale. So $(a_n^k)_{n \geq 0}$ converges in L_2 . Denote the limit still by a^k , then $\mathbb{E}_n a^k = a_n^k$. If $\nu_k \geq n$, then $a_n^k = 0$, and $\|s(a^k)\|_\infty \leq \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}}$. Thus we conclude that a^k is really a $(1, p, \infty)$ -atom. Since $\{\nu_k < \infty\} = \{s(f) > 2^k\}$, we get for $0 < q < \infty$

$$\begin{aligned} \left(\sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} &= 3 \left(\sum_{k \in \mathbb{Z}} 2^{kq} \mathbb{P}(\nu_k < \infty)^{\frac{q}{p}} \right)^{\frac{1}{q}} = 3 \left(\sum_{k \in \mathbb{Z}} 2^{kq} \mathbb{P}(s(f) > 2^k)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &\leq C \left(\sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} dy \mathbb{P}(s(f) > 2^k)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &\leq C \left(\sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} \mathbb{P}(s(f) > y)^{\frac{q}{p}} dy \right)^{\frac{1}{q}} \\ &\leq C \left(\int_0^\infty y^{q-1} \mathbb{P}(s(f) > y)^{\frac{q}{p}} dy \right)^{\frac{1}{q}} \\ &\leq C \|s(f)\|_{p,q} = C \|f\|_{H_{p,q}^s}. \end{aligned}$$

For $q = \infty$,

$$\begin{aligned}
\|(\mu_k)_{k \in \mathbb{Z}}\|_\infty &= \sup_{k \in \mathbb{Z}} |\mu_k| = 3 \cdot \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \\
&= 3 \cdot \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(s(f) > 2^k)^{\frac{1}{p}} \\
&\leq C \|s(f)\|_{p, \infty} = C \|f\|_{H_{p, \infty}^s}.
\end{aligned}$$

Consequently, $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C \|f\|_{H_{p, q}^s}$.

Conversely, if the martingale f has the above decomposition, then for an arbitrary integer k_0 , let

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k = g_n + h_n, \quad (n \in \mathbb{N}),$$

where $g_n = \sum_{k=-\infty}^{k_0-1} \mu_k a_n^k$ and $h_n = \sum_{k=k_0}^{\infty} \mu_k a_n^k$. By the sublinearity of the operator s , we have $s(f) \leq s(g) + s(h)$. Then

$$\begin{aligned}
\|s(g)\|_\infty &\leq \left\| \sum_{k=-\infty}^{k_0-1} |\mu_k| s(a^k) \right\|_\infty \leq \sum_{k=-\infty}^{k_0-1} |\mu_k| \|s(a^k)\|_\infty \\
&\leq \sum_{k=-\infty}^{k_0-1} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \\
&\leq \sum_{k=-\infty}^{k_0-1} A \cdot 2^k = A \cdot 2^{k_0}.
\end{aligned}$$

Since $s(a^k) = 0$ on the set $\{\nu_k = \infty\}$, we have $\{s(a^k) > 0\} \subset \{\nu_k < \infty\}$. Then it follows from $s(h) \leq \sum_{k=k_0}^{\infty} |\mu_k| s(a^k)$ that

$$\{s(h) > 0\} \subset \bigcup_{k=k_0}^{\infty} \{s(a^k) > 0\} \subset \bigcup_{k=k_0}^{\infty} \{\nu_k < \infty\}.$$

Then for each $0 < \varepsilon < 1$, we obtain

$$\begin{aligned}
2^{k_0 \varepsilon p} \mathbb{P}(s(h) > 2^{k_0}) &\leq 2^{k_0 \varepsilon p} \mathbb{P}(s(h) > 0) \leq 2^{k_0 \varepsilon p} \sum_{k=k_0}^{\infty} \mathbb{P}(\nu_k < \infty) \\
&= 2^{k_0 \varepsilon p} \sum_{k=k_0}^{\infty} 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) 2^{-k \varepsilon p} \\
&\leq \sum_{k=k_0}^{\infty} 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) = \sum_{k=k_0}^{\infty} \left(2^{k \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \right)^p.
\end{aligned}$$

By Lemma 2.3, we have $s(f) \in L_{p,q}$ and $\|s(f)\|_{p,q} \leq C\|\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}}\|_{l_q} \leq C\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q}$. Then $f \in H_{p,q}^s$ and $\|f\|_{H_{p,q}^s} \leq C\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q}$. Thus

$$\|f\|_{H_{p,q}^s} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q},$$

where the infimum is taken over all the above decompositions. The proof of the theorem is complete.

Remark 3.2. If $q \neq \infty$, then (3.1) holds in $H_{p,q}^s$. Namely, the sum $\sum_{k=m}^n \mu_k a^k$ converges to f in $H_{p,q}^s$ as $m \rightarrow -\infty$, $n \rightarrow \infty$. Indeed,

$$\sum_{k=m}^n \mu_k a^k = \sum_{k=m}^n (f^{\nu_{k+1}} - f^{\nu_k}) = f^{\nu_{n+1}} - f^{\nu_m}.$$

By the sublinearity of $s(f)$ we have

$$\begin{aligned} \|f - \sum_{k=m}^n \mu_k a^k\|_{H_{p,q}^s} &= \|s(f - f^{\nu_{n+1}} + f^{\nu_m})\|_{p,q} \leq \|s(f - f^{\nu_{n+1}}) + s(f^{\nu_m})\|_{p,q} \\ &\leq C \left(\|s(f - f^{\nu_{n+1}})\|_{p,q} + \|s(f^{\nu_m})\|_{p,q} \right). \end{aligned}$$

Since $s(f - f^{\nu_{n+1}})^2 = s(f)^2 - s(f^{\nu_{n+1}})^2$, then $s(f - f^{\nu_{n+1}}) \leq s(f)$, $s(f^{\nu_m}) \leq s(f)$ and $s(f - f^{\nu_{n+1}}), s(f^{\nu_m}) \rightarrow 0$ a.e. as $m \rightarrow -\infty$, $n \rightarrow \infty$. Thus by the Lebesgue convergence theorem, we have

$$\|s(f - f^{\nu_{n+1}})\|_{p,q}, \|s(f^{\nu_m})\|_{p,q} \rightarrow 0 \quad \text{as } m \rightarrow -\infty, n \rightarrow \infty,$$

which means $\|f - \sum_{k=m}^n \mu_k a^k\|_{H_{p,q}^s} \rightarrow 0$ as $m \rightarrow -\infty$, $n \rightarrow \infty$. Further, for each $k \in \mathbb{Z}$, $a^k = (a_n^k)_{n \geq 0}$ is L_2 bounded, hence $H_2^s = L_2$ is dense in $H_{p,q}^s$.

Theorem 3.3. In Theorem 3.1, if we replace $H_{p,q}^s$, $(1, p, \infty)$ -atoms by $Q_{p,q}$, $(2, p, \infty)$ -atoms (or $D_{p,q}$, $(3, p, \infty)$ -atoms) respectively, then the conclusions still hold.

Proof. The proof is similar to the one of Theorem 3.1, so we give it in sketch, only. If $f = (f_n)_{n \geq 0} \in Q_{p,q}$ (or $D_{p,q}$). The stopping times ν_k are defined in these cases by

$$\nu_k = \inf\{n \in \mathbb{N} : \lambda_n > 2^k\}, \quad (\inf \emptyset = \infty),$$

where $(\lambda_n)_{n \geq 0}$ is the sequence in the definition of $Q_{p,q}$ (or $D_{p,q}$). Let a^k and μ_k ($k \in \mathbb{Z}$) be defined as in the proof of Theorem 3.1. Then the conclusions $f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k$ ($n \in \mathbb{N}$) and $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C\|f\|_{Q_{p,q}}$ (or $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C\|f\|_{D_{p,q}}$) still hold.

To prove the converse part, let

$$\lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|S(a^k)\|_{\infty} \quad (\text{or } \lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|(a^k)^*\|_{\infty}).$$

Then $(\lambda_n)_{n \geq 0}$ is a nondecreasing, nonnegative and adapted sequence with $S_{n+1}(f) \leq \lambda_n$ (or $|f_{n+1}| \leq \lambda_n$).

For any given integer k_0 , let

$$\lambda_\infty = \lambda_\infty^{(1)} + \lambda_\infty^{(2)},$$

where

$$\lambda_\infty^{(1)} = \sum_{k=\infty}^{k_0-1} \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \text{ (or } \lambda_\infty^{(1)} = \sum_{k=\infty}^{k_0-1} \chi_{\{\nu_k < \infty\}} \|(a^k)^*\|_\infty),$$

and

$$\lambda_\infty^{(2)} = \sum_{k=k_0}^{\infty} \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \text{ (or } \lambda_\infty^{(2)} = \sum_{k=k_0}^{\infty} \chi_{\{\nu_k < \infty\}} \|(a^k)^*\|_\infty).$$

Replacing $s(g)$ and $s(h)$ in the proof of Theorem 3.1 by $\lambda_\infty^{(1)}$ and $\lambda_\infty^{(2)}$. Using Lemma 2.3, we can obtain $f \in Q_{p,q}$ (or $f \in D_{p,q}$) and $\|f\|_{Q_{p,q}} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q}$ (or $\|f\|_{D_{p,q}} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q}$), where the infimum is taken over all the above decompositions. The proof is complete.

4 Duality results

In this section, we prove the predual of the generalized BMO martingale spaces.

Theorem 4.1. *The dual space of $H_{p,q}^s$ is $BMO_2(\alpha)$, $(0 < p, q \leq 1, \alpha = \frac{1}{p} - 1)$.*

Proof. Since $0 < p, q \leq 1$, we note that by

$$\|f\|_{H_{p,q}^s} = \|s(f)\|_{p,q} \leq \|s(f)\|_{2,2} = \|f\|_2,$$

the space L_2 is a subspace of $H_{p,q}^s$. By the Remark 3.2, we know that L_2 is dense in $H_{p,q}^s$. For any $g \in BMO_2(\alpha) \subset L_2$, we show that

$$\varphi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_2,$$

is a continuous linear functional on L_2 . It follows from Theorem 3.1 that $f = \sum_{k \in \mathbb{Z}} \mu_k a^k$.

Hence

$$\varphi_g(f) = \mathbb{E}(fg) = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g).$$

By the definition of the atom a^k , we have $\mathbb{E}(a^k g) = \mathbb{E}(a^k(g - g^{\nu_k}))$. Using Hölder's inequality, we obtain

$$\begin{aligned}
|\varphi_g(f)| &= \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k(g - g^{\nu_k})) \right| \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(|a^k(g - g^{\nu_k})|) \\
&\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_2 \|g - g^{\nu_k}\|_2 = \sum_{k \in \mathbb{Z}} |\mu_k| \|s(a^k)\|_2 \|g - g^{\nu_k}\|_2 \\
&= \sum_{k \in \mathbb{Z}} |\mu_k| \|s(a^k) \chi_{\{\nu_k < \infty\}}\|_2 \|g - g^{\nu_k}\|_2 \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|s(a^k)\|_\infty \|\chi_{\{\nu_k < \infty\}}\|_2 \|g - g^{\nu_k}\|_2 \\
&\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{\frac{1}{2} - \frac{1}{p}} \|g - g^{\nu_k}\|_2 \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|g\|_{BMO_2(\alpha)}.
\end{aligned}$$

Since $0 < q \leq 1$, we have $|\varphi_g(f)| \leq \left(\sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} \|g\|_{BMO_2(\alpha)}$, and by Theorem 3.1, we obtain

$$|\varphi_g(f)| \leq C \|f\|_{H_{p,q}^s} \|g\|_{BMO_2(\alpha)}.$$

By density of L_2 in $H_{p,q}^s$, φ_g can be uniquely extended to a continuous functional on $H_{p,q}^s$.

Conversely, for any $\varphi \in (H_{p,q}^s)^*$, we show that there exists $g \in BMO_2(\alpha)$ such that $\varphi = \varphi_g$ and $\|g\|_{BMO_2(\alpha)} \leq \|\varphi\|$.

Since L_2 can be continuously embedded in $H_{p,q}^s$, then there exists $g \in L_2$ such that $\varphi(f) = \mathbb{E}(fg)$, ($f \in L_2$).

Let ν be an arbitrary stopping time and

$$h = \frac{g - g^\nu}{\|g - g^\nu\|_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}}.$$

Then $s(h) = 0$ on $\{\nu = \infty\}$, namely, $s(h) = s(h) \chi_{\{\nu < \infty\}}$.

Since $0 < p, q \leq 1$, then there exists $p_1, q_1 > 0$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{2} + \frac{1}{q_1}$.

By Hölder's inequality we have

$$\begin{aligned}
\|h\|_{H_{p,q}^s} &= \frac{\|g - g^\nu\|_{H_{p,q}^s}}{\|g - g^\nu\|_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} = \frac{\|s(g - g^\nu) \chi_{\{\nu < \infty\}}\|_{p,q}}{\|g - g^\nu\|_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} \\
&\leq \frac{C}{\|g - g^\nu\|_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} \|s(g - g^\nu)\|_{2,2} \|\chi_{\{\nu < \infty\}}\|_{p_1,q_1} \\
&= \frac{C \|g - g^\nu\|_2}{\|g - g^\nu\|_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} \left(\frac{q_1}{p_1} \int_0^\infty t^{\frac{q_1}{p_1} - 1} \left(\mu_t(\chi_{\{\nu < \infty\}}) \right)^{q_1} dt \right)^{\frac{1}{q_1}} \\
&= \frac{C}{\mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} \left(\frac{q_1}{p_1} \int_0^\infty t^{\frac{q_1}{p_1} - 1} \chi_{[0, \mathbb{P}(\nu < \infty))}^{q_1}(t) dt \right)^{\frac{1}{q_1}} \\
&= \frac{C}{P(\nu < \infty)^{\frac{1}{p} - \frac{1}{2}}} \mathbb{P}(\nu < \infty)^{\frac{1}{p_1}} = C.
\end{aligned}$$

Set $h_0 = h/C$, then $\|h_0\|_{H_{p,q}^s} \leq 1$. Consequently, $\|\varphi\| \geq |\varphi(h_0)| = \mathbb{E}(h_0 g) = \mathbb{E}(h_0(g - g^\nu)) = C^{-1} \mathbb{P}(\nu < \infty)^{\frac{1}{2} - \frac{1}{p}} \|g - g^\nu\|_2$. Taking the supremum over all stopping times, we have $\|g\|_{BMO_2(\alpha)} \leq C \|\varphi\|$. The proof of the theorem is complete.

Now we investigate the case $0 < p \leq 1, 1 < q < \infty$.

Theorem 4.2. *The dual space of $H_{p,q}^s$ is $BMO_{2,q}(\alpha)$, ($0 < p \leq 1, 1 < q < \infty, \alpha = \frac{1}{p} - 1$).*

Proof. Let $g \in BMO_{2,q}(\alpha) \subset L_2$, define $\varphi_g(f) = \mathbb{E}(fg)$, ($f \in L_2$). Similarly to the proof of Theorem 4.1, by Hölder's inequality we have

$$\begin{aligned} |\varphi_g(f)| &= \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^{\nu_k})) \right| \\ &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(|a^k (g - g^{\nu_k})|) \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_2 \|g - g^{\nu_k}\|_2 \\ &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{\frac{1}{2} - \frac{1}{p}} \|g - g^{\nu_k}\|_2 = A \sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \|g - g^{\nu_k}\|_2. \end{aligned}$$

By the definition of $\|\cdot\|_{BMO_{2,q}(\alpha)}$ and Theorem 3.1, then

$$|\varphi_g(f)| \leq A \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|g\|_{BMO_{2,q}(\alpha)} \leq C \|f\|_{H_{p,q}^s} \|g\|_{BMO_{2,q}(\alpha)}.$$

Thus φ_g can be uniquely extended to a continuous functional on $H_{p,q}^s$.

Conversely, if $\varphi \in (H_{p,q}^s)^*$, we know that there exists $g \in L_2$ such that $\varphi(f) = \mathbb{E}(fg)$, ($f \in L_2$). Let $\{\nu_k\}_{k \in \mathbb{Z}}$ be an arbitrary stopping time sequence such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$ and N be an arbitrary nonnegative integer. Let

$$h_k = \frac{|g - g^{\nu_k}| \text{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_2}, \quad f = \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} (h_k - h_k^{\nu_k}).$$

For an arbitrary integer k_0 which satisfies $-N \leq k_0 \leq N$ (for $k_0 \leq -N$, let $G = 0$ and $H = f$; for $k_0 > N$, let $H = 0$ and $G = f$), let

$$f = G + H,$$

where $G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} (h_k - h_k^{\nu_k})$ and $H = \sum_{k=k_0}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} (h_k - h_k^{\nu_k})$.

Obviously $\|h_k\|_2 = 1$, and $\|G\|_2 \leq 2 \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}}$. By the sublinearity of the

operator s , we have $s(f) \leq s(G) + s(H)$. Let $\varepsilon = \frac{p}{2}$, then $0 < \varepsilon < \min(1, \frac{q}{p})$. We obtain

$$\begin{aligned}
2^{k_0 p} \mathbb{P}(s(G) > 2^{k_0})^\varepsilon &\leq 2^{k_0 p} \left(\frac{1}{2^{2k_0}} \|s(G)\|_2^2 \right)^\varepsilon \leq C \cdot 2^{k_0(p-2\varepsilon)} \|G\|_2^{2\varepsilon} \\
&\leq C \left(\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \right)^{2\varepsilon} = C \left(\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}} \right)^p \\
&\leq C \sum_{k=-N}^{k_0-1} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}})^p \leq C \sum_{k=-\infty}^{k_0-1} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}})^p.
\end{aligned}$$

On the other hand,

$$\{s(H) > 0\} \subset \bigcup_{k=k_0}^N \{\nu_k < \infty\}.$$

Then for each $0 < \varepsilon < 1$, we have

$$\begin{aligned}
2^{k_0 \varepsilon p} \mathbb{P}(s(H) > 2^{k_0}) &\leq 2^{k_0 \varepsilon p} \mathbb{P}(s(H) > 0) \leq 2^{k_0 \varepsilon p} \sum_{k=k_0}^N \mathbb{P}(\nu_k < \infty) \\
&= 2^{k_0 \varepsilon p} \sum_{k=k_0}^N 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) 2^{-k \varepsilon p} \leq \sum_{k=k_0}^N 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) \\
&= \sum_{k=k_0}^N (2^{k \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p \leq \sum_{k=k_0}^{\infty} (2^{k \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p.
\end{aligned}$$

By Lemma 2.4, we have $s(f) \in L_{p,q}$ and $\|s(f)\|_{p,q} \leq C \|\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}}\|_{l_q}$. Thus $f \in H_{p,q}^s$ and

$$\|f\|_{H_{p,q}^s} \leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned}
\sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \|g - g^{\nu_k}\|_2 &= \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}(h_k(g - g^{\nu_k})) \\
&= \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}((h_k - h_k^{\nu_k})g) \\
&= \mathbb{E}(fg) = \varphi(f) \leq \|f\|_{H_{p,q}^s} \|\varphi\| \\
&\leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|\varphi\|.
\end{aligned}$$

Thus we obtain

$$\frac{\sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \|g - g^{\nu_k}\|_2}{\left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}} \leq C \|\varphi\|.$$

Taking over all $N \in \mathbb{N}$ and the supremum over all of such stopping time sequences satisfying $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$, we get $\|g\|_{BMO_{2,q}(\alpha)} \leq C\|\varphi\|$. The proof is complete.

5 The generalized John-Nirenberg theorem

In this section, we prove the generalized John-Nirenberg theorem by duality when the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Some of the dual results are of independent interest. In order to do this, we need the following lemma and we refer to [22] for these facts.

Lemma 5.1. *If the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, then the martingale Hardy-Lorentz spaces $H_{p,q}^*$, $H_{p,q}^S$, $H_{p,q}^s$, $Q_{p,q}$ and $D_{p,q}$ are all equivalent for $0 < p < \infty, 0 < q \leq \infty$, and $H_{p,q}^*$, $H_{p,q}^S$, $H_{p,q}^s$, $Q_{p,q}$, $D_{p,q}$ and $L_{p,q}$ are all equivalent for $1 < p < \infty, 0 < q \leq \infty$.*

Theorem 5.2. *If the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, then*

$$(H_{p,q}^s)^* = BMO_{r,q}(\alpha), \quad (0 < p \leq 1, 1 < q, r < \infty, \alpha = \frac{1}{p} - 1).$$

Proof. Let $g \in BMO_{r,q}(\alpha) \subset L_r$ and r' be the conjugate number of r , then $1 < r' < \infty$. Define $\varphi_g(f) = \mathbb{E}(fg)$, $f \in L_{r'}$. Note that $L_{r'} = H_{r'}^s \subset H_{p,q}^s$. By Theorem 3.1 there exists a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(1, p, \infty)$ -atoms and a sequence of real numbers $(\mu_k)_{k \in \mathbb{Z}}$ satisfying $\mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}$ (where A is a positive constant and $(\nu_k)_{k \in \mathbb{Z}}$ is the corresponding stopping time sequence) such that $f = \sum_{k \in \mathbb{Z}} \mu_k a^k$ and $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C\|f\|_{H_{p,q}^s}$. By Hölder's inequality we can obtain

$$\begin{aligned} |\varphi_g(f)| &= \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^{\nu_k})) \right| \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(|a^k (g - g^{\nu_k})|) \\ &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_{r'} \|g - g^{\nu_k}\|_r \leq C \sum_{k \in \mathbb{Z}} |\mu_k| \|s(a^k)\|_{r'} \|g - g^{\nu_k}\|_r \\ &\leq C \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'} - \frac{1}{p}} \|g - g^{\nu_k}\|_r = C \cdot A \sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1 - \frac{1}{r}} \|g - g^{\nu_k}\|_r. \end{aligned}$$

By the definition of $\|\cdot\|_{BMO_{r,q}(\alpha)}$, we obtain

$$|\varphi_g(f)| \leq C \cdot A \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|g\|_{BMO_{r,q}(\alpha)} \leq C \|f\|_{H_{p,q}^s} \|g\|_{BMO_{r,q}(\alpha)}.$$

Thus φ_g can be extended to a continuous functional on $H_{p,q}^s$.

Conversely, if $\varphi \in (H_{p,q}^s)^*$. By the regularity of the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$, we have $L_{r'} = H_{r',r'}^s \subset H_{p,q}^s$, then $(H_{p,q}^s)^* \subset (L_{r'})^* = L_r$. Thus there exists $g \in L_r$ such that $\varphi(f) = \varphi_g(f) = \mathbb{E}(fg)$, ($f \in L_{r'}$).

Let $\{\nu_k\}_{k \in \mathbb{Z}}$ be an arbitrary stopping time sequence such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$ and N be an arbitrary nonnegative integer. Let

$$h_k = \frac{|g - g^{\nu_k}|^{r-1} \text{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_r^{r-1}}, \quad f = \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} (h_k - h_k^{\nu_k}).$$

For an arbitrary integer k_0 which satisfies $-N \leq k_0 \leq N$ (for $k_0 \leq -N$, let $G = 0$ and $H = f$; for $k_0 > N$, let $H = 0$ and $G = f$), let

$$f = G + H,$$

where $G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} (h_k - h_k^{\nu_k})$ and $H = \sum_{k=k_0}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} (h_k - h_k^{\nu_k})$.

Obviously, $\|h_k\|_{r'} = 1$ and $\|G\|_{r'} \leq 2 \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}}$. By the sublinearity of the operator s , we have $s(f) \leq s(G) + s(H)$. Let $\varepsilon = \frac{p}{r'}$, then $0 < \varepsilon < \min(1, \frac{q}{p})$. By Lemma 5.1 we have

$$\begin{aligned} 2^{k_0 p} \mathbb{P}(s(G) > 2^{k_0})^\varepsilon &\leq 2^{k_0 p} \left(\frac{1}{2^{k_0 r'}} \|s(G)\|_{r'}^{r'} \right)^\varepsilon \leq C \cdot 2^{k_0(p-r'\varepsilon)} \|G\|_{r'}^{r'\varepsilon} \\ &\leq C \left(\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} \right)^{r'\varepsilon} = C \left(\sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}} \right)^p \\ &\leq C \sum_{k=-N}^{k_0-1} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}})^p \leq C \sum_{k=-\infty}^{k_0-1} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{\varepsilon}{p}})^p. \end{aligned}$$

On the other hand, $\{s(H) > 0\} \subset \bigcup_{k=k_0}^N \{\nu_k < \infty\}$. Then for each $0 < \varepsilon < 1$, we have

$$\begin{aligned} 2^{k_0 \varepsilon p} \mathbb{P}(s(H) > 2^{k_0}) &\leq 2^{k_0 \varepsilon p} \mathbb{P}(s(H) > 0) \leq 2^{k_0 \varepsilon p} \sum_{k=k_0}^N \mathbb{P}(\nu_k < \infty) \\ &\leq \sum_{k=k_0}^N 2^{k \varepsilon p} \mathbb{P}(\nu_k < \infty) = \sum_{k=k_0}^N (2^{k \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p \\ &\leq \sum_{k=k_0}^{\infty} (2^{k \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p. \end{aligned}$$

By Lemma 2.4, we have $s(f) \in L_{p,q}$ and $\|s(f)\|_{p,q} \leq C \|\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}}\|_{l_q}$. Thus $f \in H_{p,q}^s$ and

$$\|f\|_{H_{p,q}^s} \leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}.$$

Consequently,

$$\begin{aligned}
\sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{r}} \|g - g^{\nu_k}\|_r &= \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} \mathbb{E}(h_k(g - g^{\nu_k})) \\
&= \sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{r'}} \mathbb{E}((h_k - h_k^{\nu_k})g) \\
&= \mathbb{E}(fg) = \varphi(f) \leq \|f\|_{H_{p,q}^s} \|\varphi\| \\
&\leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|\varphi\|.
\end{aligned}$$

Thus we obtain

$$\frac{\sum_{k=-N}^N 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{r}} \|g - g^{\nu_k}\|_r}{\left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}} \leq C \|\varphi\|.$$

Taking $N \rightarrow \infty$ and the supremum over all of such stopping time sequences such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$, we get $\|g\|_{BMO_{r,q}(\alpha)} \leq C \|\varphi\|$. The proof is complete.

It should be mentioned that the proof method of Theorem 5.2 is not available for $r = 1$. In this case, we need new insight. Let the dual space of $D_{p,q}$ be $D_{p,q}^*$. Let us denote by $(D_{p,q}^*)_1$ those elements φ from $D_{p,q}^*$ for which there exists $g \in L_1$ such that $\varphi(f) = \mathbb{E}(fg)$, $f \in L_\infty$. Namely

$$(D_{p,q}^*)_1 = \{\varphi \in D_{p,q}^* : \exists g \in L_1 \text{ s.t. } \varphi(f) = \mathbb{E}(fg), \forall f \in L_\infty\}.$$

Theorem 5.3. $(D_{p,q}^*)_1 = BMO_1(\alpha)$, $(0 < p, q \leq 1, \alpha = \frac{1}{p} - 1)$.

Proof. Let $g \in BMO_1(\alpha) \subset L_1$. Define $\varphi_g(f) = \mathbb{E}(fg)$, $(f \in L_\infty)$. By Theorem 3.3, there exists a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(3, p, \infty)$ -atoms and a sequence of real numbers $(\mu_k)_{k \in \mathbb{Z}}$ satisfying $\mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}$ (where A is a positive constant and $(\nu_k)_{k \in \mathbb{Z}}$ is the corresponding stopping time sequence) such that $f = \sum_{k \in \mathbb{Z}} \mu_k a^k$ and $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C \|f\|_{D_{p,q}}$. By Hölder's inequality we obtain

$$\begin{aligned}
|\varphi_g(f)| &= \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^{\nu_k})) \right| \\
&\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(|a^k (g - g^{\nu_k})|) \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_\infty \|g - g^{\nu_k}\|_1 \\
&\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|(a^k)^*\|_\infty \|g - g^{\nu_k}\|_1 \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g - g^{\nu_k}\|_1 \\
&\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|g\|_{BMO_1(\alpha)}.
\end{aligned}$$

Since $0 < q \leq 1$, then

$$|\varphi_g(f)| \leq \left(\sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} \|g\|_{BMO_1(\alpha)} \leq C \|f\|_{D_{p,q}} \|g\|_{BMO_1(\alpha)}.$$

Then φ_g can be extended to a continuous functional on $D_{p,q}$, and $\varphi_g \in (D_{p,q}^*)_1$.

To prove the converse, let $\varphi \in (D_{p,q}^*)_1$, then there exists $g \in L_1$ such that $\varphi(f) = \mathbb{E}(fg)$, ($f \in L_\infty$). Let $h = \text{sign}(g - g^\nu)$, $a = \frac{1}{2} \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} (h - h^\nu)$, where $\nu \in \mathcal{T}$ is an arbitrary stopping time. Then a is a $(3, p, \infty)$ -atom.

Let $\mu = 2A \cdot \mathbb{P}(\nu < \infty)^{\frac{1}{p}}$, let $h_0 = \mu a = A(h - h^\nu)$. Considering the atomic decomposition of h_0 , by Theorem 3.2 we have $h_0 \in D_{p,q}$ and $\|h_0\|_{D_{p,q}} \leq C|\mu| = 2CA \cdot \mathbb{P}(\nu < \infty)^{\frac{1}{p}}$, then $\|h - h^\nu\|_{D_{p,q}} \leq 2C \cdot \mathbb{P}(\nu < \infty)^{\frac{1}{p}}$. Thus we have

$$\begin{aligned} \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \|g - g^\nu\|_1 &= \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \mathbb{E}(h(g - g^\nu)) = \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \mathbb{E}((h - h^\nu)g) \\ &= \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \varphi(h - h^\nu) \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \|h - h^\nu\|_{D_{p,q}} \|\varphi\| \\ &= 2C \|\varphi\|. \end{aligned}$$

Taking the supremum over all stopping times, then we obtain $\|g\|_{BMO_1(\alpha)} \leq C \|\varphi\|$. The proof of the theorem is complete.

Now we consider $(D_{p,q}^*)_1$, ($0 < p \leq 1, 1 < q < \infty$). We have the following theorem.

Theorem 5.4. $(D_{p,q}^*)_1 = BMO_{1,q}(\alpha)$, ($0 < p \leq 1, 1 < q < \infty, \alpha = \frac{1}{p} - 1$).

Proof. Let $g \in BMO_{1,q}(\alpha) \subset L_1$, then

$$\|g\|_{BMO_{1,q}(\alpha)} = \sup \frac{\sum_{k \in \mathbb{Z}} 2^k \|g - g^{\nu_k}\|_1}{\left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}} < \infty,$$

where the supremum is taken over all stopping time sequences $\{\nu_k\}_{k \in \mathbb{Z}} \subset \mathcal{T}$ such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$. Define $\varphi_g(f) = \mathbb{E}(fg)$, ($f \in L_\infty$). Similarly to the proof of Theorem 4.3, by Hölder's inequality we can obtain

$$\begin{aligned} |\varphi_g(f)| &= \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^{\nu_k})) \right| \\ &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(|a^k (g - g^{\nu_k})|) \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_\infty \|g - g^{\nu_k}\|_1 \\ &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g - g^{\nu_k}\|_1 = A \sum_{k \in \mathbb{Z}} 2^k \|g - g^{\nu_k}\|_1. \end{aligned}$$

By the definition of $\|\cdot\|_{BMO_{1,q}(\alpha)}$, we obtain

$$|\varphi_g(f)| \leq A \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|g\|_{BMO_{1,q}(\alpha)} \leq C \|f\|_{D_{p,q}} \|g\|_{BMO_{1,q}(\alpha)}.$$

Thus φ_g can be extended to a continuous functional on $D_{p,q}$. Moreover, $\varphi_g \in (D_{p,q}^*)_1$.

Conversely, if $\varphi \in (D_{p,q}^*)_1$, then there exists $g \in L_1$ such that $\varphi(f) = \mathbb{E}(fg)$, ($f \in L_\infty$). Let $\{\nu_k\}_{k \in \mathbb{Z}}$ be an arbitrary stopping time sequence such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$. Let

$$h_k = \text{sign}(g - g^{\nu_k}), \quad a^k = \frac{1}{2}(h_k - h_k^{\nu_k})\mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}}.$$

then a^k is a $(3, p, \infty)$ -atom.

Let $f^N = \sum_{k=-N}^N 2^{k+1} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} a^k$, where N is an arbitrary nonnegative integer.

By Theorem 3.3 we have $f^N \in D_{p,q}$ and

$$\|f^N\|_{D_{p,q}} \leq C \left(\sum_{k=-N}^N (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}.$$

Consequently,

$$\begin{aligned} \sum_{k=-N}^N 2^k \|g - g^{\nu_k}\|_1 &= \sum_{k=-N}^N 2^k \mathbb{E}(h_k(g - g^{\nu_k})) = \sum_{k=-N}^N 2^k \mathbb{E}((h_k - h_k^{\nu_k})g) \\ &= \mathbb{E}(f^N g) = \varphi(f^N) \leq \|f^N\|_{D_{p,q}} \|\varphi\| \\ &\leq C \left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|\varphi\|. \end{aligned}$$

Thus we have

$$\frac{\sum_{k=-N}^N 2^k \|g - g^{\nu_k}\|_1}{\left(\sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}} \leq C \|\varphi\|.$$

This shows $\|g\|_{BMO_{1,q}(\alpha)} \leq C \|\varphi\|$. The proof is complete.

Proposition 5.5. *If the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, for $0 < p \leq 1, 0 < q < \infty$, then $(D_{p,q}^*)_1 = D_{p,q}^*$.*

Proof. Since $0 < p \leq 1$, then by Lemma 5.1, L_2 can also be embedded continuously in $D_{p,q}$. Then $D_{p,q}^* \subset (L_2)^* = L_2$. Let φ be an arbitrary element of $D_{p,q}^*$, then there exists $g \in L_2 \subset L_1$ such that $\varphi = \varphi_g$. By the definition of $(D_{p,q}^*)_1$, we have $\varphi \in (D_{p,q}^*)_1$, then $D_{p,q}^* \subset (D_{p,q}^*)_1$. And the inclusion relation $(D_{p,q}^*)_1 \subset D_{p,q}^*$ is evident. Hence we obtain

$$(D_{p,q}^*)_1 = D_{p,q}^*, \quad (0 < p \leq 1, 0 < q < \infty).$$

The proof of the proposition is complete.

We now are a position to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from Theorem 4.2 and Theorem 5.2 that

$$BMO_{r,q}(\alpha) = BMO_{2,q}(\alpha), \quad 1 < r < \infty.$$

For $r = 1$, combining Theorem 4.2, Lemma 5.1, Theorem 5.4 with Proposition 5.5, we get

$$BMO_{1,q}(\alpha) = BMO_{2,q}(\alpha).$$

6 Boundedness of fractional integrals on martingale Hardy-Lorentz spaces

As we know, Chao and Ombe [4] introduced the fractional integrals for dyadic martingales. Recently, Nakai and Sadasue [17] extended the notion of fractional integrals to more general martingales. Sadasue [19] proves the boundedness of fractional integrals on martingale Hardy spaces for $0 < p \leq 1$. We now extend the boundedness of fractional integrals to martingale Hardy-Lorentz spaces. In this section, we suppose that every σ -algebra \mathcal{F}_n is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom, if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $\mathbb{P}(A) < \mathbb{P}(B)$, then $\mathbb{P}(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in \mathcal{F}_n . Without loss of generality, we always suppose that the constant in (1.3) satisfying $R \geq 2$.

Now we give the definition of fractional integral as follows.

Definition 6.1. For $f = (f_n)_{n \geq 0} \in \mathcal{M}$, $\alpha > 0$, the fractional integral $I_\alpha f = ((I_\alpha f)_n)_{n \geq 0}$ of f is defined by

$$(I_\alpha f)_n = \sum_{k=1}^n b_{k-1}^\alpha d_k f.$$

where b_k is an \mathcal{F}_k -measurable function such that $\forall B \in A(\mathcal{F}_k), \forall \omega \in B, b_k(\omega) = \mathbb{P}(B)$.

In order to prove the boundedness of fractional integrals, we need the following lemmas.

Lemma 6.2. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular, $f \in \mathcal{M}$ and $\alpha > 0$. Let R be the constant in (1.3). If there exists $B \in \mathcal{F}$ such that $f^* \leq \chi_B$. Then there exists a positive constant C_α independent of f and B such that

$$(I_\alpha f)^* \leq C_\alpha \mathbb{P}(B)^\alpha \chi_B.$$

For the proof of Lemma 6.2, see [19], Lemma 3.5.

In the next lemma, we regard $(3, p, \infty)$ -atom a as a martingale by $a = (a_n)_{n \geq 0} = (E_n(a))_{n \geq 0}$, so we can consider the fractional integral $I_\alpha a = ((I_\alpha a)_n)_{n \geq 0}$.

Lemma 6.3. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular and R be the constant in (1.3). If $0 < p_1 < p_2 < \infty$, $\alpha = \frac{1}{p_1} - \frac{1}{p_2}$, $0 < q_2 \leq \infty$, and a is a $(3, p_1, \infty)$ -atom as in Definition 2.2. Then we have*

$$\|I_\alpha a\|_{H_{p_2, q_2}^*} \leq C_\alpha,$$

where C_α is the same constant as in Lemma 6.2.

Proof. Let ν be the stopping time associated with a . Then we have $a^* \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} \chi_{\{\nu < \infty\}}$. Therefore $(\mathbb{P}(\nu < \infty)^{\frac{1}{p_1}} a)^* = \mathbb{P}(\nu < \infty)^{\frac{1}{p_1}} a^* \leq \chi_{\{\nu < \infty\}}$. By Lemma 6.2 we can obtain $(I_\alpha(\mathbb{P}(\nu < \infty)^{\frac{1}{p_1}} a))^* \leq C_\alpha \mathbb{P}(\nu < \infty)^\alpha \chi_{\{\nu < \infty\}}$. Then

$$(I_\alpha a)^* \leq C_\alpha \mathbb{P}(\nu < \infty)^\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} \chi_{\{\nu < \infty\}} = C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{\nu < \infty\}}.$$

By Proposition 2.1, we have

$$\mu_t((I_\alpha a)^*) \leq \mu_t(C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{\nu < \infty\}}) = C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{[0, \mathbb{P}(\nu < \infty))}(t).$$

For $0 < q_2 < \infty$, then

$$\begin{aligned} \|I_\alpha a\|_{H_{p_2, q_2}^*}^{q_2} &= \|(I_\alpha a)^*\|_{p_2, q_2}^{q_2} = \frac{q_2}{p_2} \int_0^\infty t^{\frac{q_2}{p_2}-1} \left(\mu_t((I_\alpha a)^*) \right)^{q_2} dt \\ &\leq \frac{q_2}{p_2} \int_0^\infty t^{\frac{q_2}{p_2}-1} \left(C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{[0, \mathbb{P}(\nu < \infty))}(t) \right)^{q_2} dt \\ &= \frac{q_2}{p_2} \int_0^{\mathbb{P}(\nu < \infty)} t^{\frac{q_2}{p_2}-1} C_\alpha^{q_2} \mathbb{P}(\nu < \infty)^{-\frac{q_2}{p_2}} dt \\ &= C_\alpha^{q_2}. \end{aligned}$$

For $q_2 = \infty$, then

$$\begin{aligned} \|I_\alpha a\|_{H_{p_2, \infty}^*} &= \|(I_\alpha a)^*\|_{p_2, \infty} = \sup_{t > 0} t^{\frac{1}{p_2}} \mu_t((I_\alpha a)^*) \\ &\leq \sup_{t > 0} C_\alpha t^{\frac{1}{p_2}} \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{[0, \mathbb{P}(\nu < \infty))}(t) \\ &= C_\alpha. \end{aligned}$$

Therefore $\|I_\alpha a\|_{H_{p_2, q_2}^*} \leq C_\alpha$, where C_α is the same constant as in Lemma 6.2. The proof of is complete.

Theorem 6.4. *Let (Ω, \mathcal{F}, P) be a complete and nonatomic probability space, and $\{\mathcal{F}_n\}_{n \geq 0}$ be a regular stochastic basis, let $0 < q_1 \leq 1, q_1 \leq q_2, q_1 \leq p_2, 0 < p_1 < p_2 < \infty$, $\alpha = \frac{1}{p_1} - \frac{1}{p_2}$, then there exists a constant C such that*

$$\|I_\alpha f\|_{H_{p_2, q_2}^*} \leq C \|f\|_{H_{p_1, q_1}^*},$$

for all $f \in H_{p_1, q_1}^*$.

Proof. For $f \in H_{p_1, q_1}^*$. Since $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, by Theorem 3.3 and Lemma 5.1, there exists a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(3, p_1, \infty)$ -atoms and a real number sequence $(\mu_k)_{k \in \mathbb{Z}} \in l_{q_1}$ such that

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k, \quad (n \in \mathbb{N}),$$

and

$$\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_{q_1}} \leq C \|f\|_{H_{p_1, q_1}^*}.$$

Then by Lemma 6.3, we have

$$\begin{aligned} \|I_\alpha f\|_{H_{p_2, q_2}^*}^{q_1} &= \|(I_\alpha f)^*\|_{p_2, q_2}^{q_1} = \|(I_\alpha (\sum_{k \in \mathbb{Z}} \mu_k a^k))^*\|_{p_2, q_2}^{q_1} \\ &\leq \|\sum_{k \in \mathbb{Z}} |\mu_k| (I_\alpha a^k)^*\|_{p_2, q_2}^{q_1} \leq C \|\sum_{k \in \mathbb{Z}} |\mu_k| (I_\alpha a^k)^*\|_{p_2, q_1}^{q_1} \\ &\leq C \sum_{k \in \mathbb{Z}} |\mu_k|^{q_1} \|(I_\alpha a^k)^*\|_{p_2, q_1}^{q_1} \leq C \cdot C_\alpha^{q_1} \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_{q_1}}^{q_1} \\ &\leq C \|f\|_{H_{p_1, q_1}^*}^{q_1}. \end{aligned}$$

Thus we have

$$\|I_\alpha f\|_{H_{p_2, q_2}^*} \leq C \|f\|_{H_{p_1, q_1}^*}.$$

The proof of the theorem is complete.

Remark 6.5. In Theorem 6.4, if we consider the special case $p_1 = q_1 = p, p_2 = q_2 = q$, then we obtain the boundedness of fractional integrals on martingale Hardy spaces for $0 < p \leq 1$, Theorem 3.1 in [19] due to Sadasue.

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